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2004 J. Phys. A: Math. Gen. 37 8017

(http://iopscience.iop.org/0305-4470/37/33/004)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 8017–8030 PII: S0305-4470(04)73054-1

Soliton on a cnoidal wave background in the coupled nonlinear Schrödinger equation

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Received 5 December 2003
Published 4 August 2004
Online at stacks.iop.org/JPhysA/37/8017
doi:10.1088/0305-4470/37/33/004

Abstract

An application of the Darboux transformation on a cnoidal wave background in the coupled nonlinear Schrödinger equation gives a new solution which describes a soliton moving on a cnoidal wave. This is a generalized version of the previously known soliton solutions of dark–bright pair. Here a dark soliton resides on a cnoidal wave instead of on a constant background. It also exhibits a new type of soliton solution in a self-focusing medium, which describes a breakup of a generalized dark–bright pair into another generalized dark–bright pair and an 'oscillating' soliton. We calculate the shift of the crest of the cnoidal wave along a soliton and the moving direction of the soliton on a cnoidal wave.

PACS numbers: 05.45.Yv, 05.60.Gg, 42.65.Tg

1. Introduction

In this paper, we consider a system of coupled nonlinear Schrödinger (CNLS) equations [1–3]

$$\partial_{\bar{z}}\psi_k = -\mathrm{i}\partial_z^2\psi_k - 2\mathrm{i}\sigma(|\psi_1|^2 + |\psi_2|^2)\psi_k \qquad k = 1, 2. \tag{1}$$

These equations, which were named as the Manakov model, are important for a number of physical applications like multi-frequency and/or two different polarizations of light in a fibre [4–6]. Nowadays, there exists a vast amount of exact solutions of this system including exact vector solitons [1, 5], bound solitary waves [7] and periodic solutions [7–10]. Especially, quasi-periodic solutions in terms of N-phase theta functions for the Manakov model are derived in [11], while a series of special solutions is given in [12–15].

Recently, this equation has attracted new attention as it can describe the localized states in optically induced refractive index gratings [16–18]. Strong incoherent interaction of such a grating with a probe beam facilitates the formation of a noble type of a composite optical soliton, where one of the components (described by ψ_1 in equation (1)) creates periodic photonic structure, while the other component (ψ_2) experiences Bragg reflection from this structure and forms gap solitons. Optically induced lattices are a very exciting development which opens up creating dynamically reconfigurable photonic structures. It was shown that the CNLS equation describes the propagation of two incoherently interacting beams in a photorefractive crystal in the limit of weak saturation regime [18, 19].

To analyse the behaviour of the soliton in the optically induced lattices, we need a solution of the CNLS equation that describes a soliton moving on a cnoidal wave. The integrability of the CNLS equation shown by Manakov provides us with the opportunity to obtain various types of solutions. Especially, one can obtain the bright soliton in a focusing medium by applying the inverse scattering method (ISM) [20]. On the other hand, in the case of nonvanishing background fields, the inverse scattering method is technically highly involved and only the dark solitons of the simplest one-component case have been found in this way. Instead, the Hirota method has been adopted to obtain *n*-bright solitons on a continuous wave background of the one-component case, as well as dark solitons of the two-component case [21–23]. The Hirota method, however, does not provide a way to construct a soliton solution moving on a cnoidal wave background. Interestingly, the (soliton+cnoidal wave) solution can be obtained from the general quasi-periodic solutions of N-phase theta functions, by taking the degenerate limit of the two-phase solution. In fact, [24] applied this procedure to obtain a solution of the single-component nonlinear Schrödinger equation (NLSE). But solutions from the *N*-phase theta functions have the so-called 'effectivization' problem, which is related to extracting the physical solutions by taking proper initial conditions [25, 10]. Much more, these solutions have a rather complicated form, which makes them difficult to apply to real situations.

In this paper, we employ a simple, but very powerful soliton finding technique based on the Darboux transformation (DT). This method is essentially equivalent to the ISM, but avoids mathematical technicalities of the ISM. Section 2 introduces the main method including the DT and Sym's solution. Section 3 analyses the characteristics of obtained solutions in the case of focusing medium. Section 4 is devoted to the case of defocusing medium. Section 5 contains a short discussion and the appendices give some proofs of equations in the main text.

2. The method

2.1. Lax pair

We first bring the CNLS equation into a matrix form in terms of 3×3 matrices E, T and $\tilde{E} = [T, E]$,

$$E = \begin{pmatrix} 0 & \psi_1 & \psi_2 \\ -\sigma\psi_1^* & 0 & 0 \\ -\sigma\psi_2^* & 0 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} \mathrm{i}/2 & 0 & 0 \\ 0 & -\mathrm{i}/2 & 0 \\ 0 & 0 & -\mathrm{i}/2 \end{pmatrix}, \tag{2}$$

such that

$$\partial_{\bar{z}} E = -\partial_z^2 \tilde{E} + 2E^2 \tilde{E}. \tag{3}$$

One can readily check that the components of equation (3) are indeed equivalent to the Manakov equation in equation (1). The signature σ is either 1 or -1 depending on whether the group velocity dispersion is abnormal ($\sigma = 1$) or normal ($\sigma = -1$), or the waveguide is self-focusing ($\sigma = 1$) or self-defocusing ($\sigma = -1$). One advantage of using matrices is that we can write down the associated linear equation (Lax pair);

$$(\partial_z + E + \lambda T)\Psi = 0 \qquad (\partial_{\bar{z}} + E\tilde{E} - \partial_z \tilde{E} - \lambda E - \lambda^2 T)\Psi = 0, \tag{4}$$

where λ is an arbitrary complex number and $\Psi(z, \bar{z}, \lambda)$ is a three-component vector.

2.2. Darboux transformation

The following cnoidal wave solution of the CNLS equation properly describes the background periodic wave in optically induced refractive index gratings. For the case of focusing medium ($\sigma = 1$), it is

$$\psi_1^c(z, \bar{z}) = p\,\mathrm{dn}(\chi, k)\,\mathrm{e}^{\mathrm{i}\zeta} \qquad \psi_2^c = 0, \tag{5}$$

where $\chi = -p(z - v\bar{z})$, $\zeta = [-vz/2 - p^2(2 - k^2)\bar{z} + v^2\bar{z}/4]$. The defocusing case ($\sigma = -1$) is obtained by substituting $p \to \mathrm{i}p$, $k \to \mathrm{i}k$ in equation (5) of the focusing case[1]. Here dn, sn is the standard Jacobi elliptic function. And v is the velocity of the cnoidal wave and $k \in (0, 1)$ is the modulus of the Jacobi function. As far as elliptic functions are involved, we employ terminology and notation of [26] without further explanations. To describe the characteristics of the composite optical soliton formed on the background grating, we need a soliton solution superposed on the cnoidal wave. To obtain a superposed solution of (soliton+cnoidal wave) using the Darboux transformation, we need to find a solution of the linear equations (4) with ψ_i, $i = 1, 2$, given by equation (5). We denote the solution of the linear equation as a three-component vector, $\Psi = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix}$. Then a new solution of (soliton+cnoidal wave) is constructed using the Darboux transformation [27–29], which is

$$\psi_i^{c-s}(z, \bar{z}) = \psi_i^c(z, \bar{z}) + \mathrm{i}(\lambda - \lambda^*)\frac{\sigma s_0 s_i^*}{|s_0|^2 + \sigma \sum_{j=1,2} |s_j|^2} \qquad i = 1, 2. \tag{6}$$

Using that s_i satisfy the associated linear equations in equation (4), it can be directly checked that ψ_i^{c-s}, $i = 1, 2$, in equation (6) is a new solution of the CNLS equation.

2.3. Sym's solution

Explicitly, the solution of the linear equations (4) for $\sigma = 1$ can be written down as

$$\begin{aligned} s_0 &= \mathrm{e}^{\mathrm{i}\zeta/2}\left[M\,\mathrm{e}^{\mathrm{i}\Delta}\theta_2\left(\frac{-\mathrm{i}u}{2K}\right)\theta_0\left(\frac{\chi + \mathrm{i}u}{2K}\right) - N\,\mathrm{e}^{-\mathrm{i}\Delta}\theta_1\left(\frac{-\mathrm{i}u}{2K}\right)\theta_3\left(\frac{\chi - \mathrm{i}u}{2K}\right)\right]\Big/\theta_0\left(\frac{\chi}{2K}\right), \\ s_1 &= \mathrm{e}^{-\mathrm{i}\zeta/2}\left[-M\,\mathrm{e}^{\mathrm{i}\Delta}\theta_1\left(\frac{-\mathrm{i}u}{2K}\right)\theta_3\left(\frac{\chi + \mathrm{i}u}{2K}\right) + N\,\mathrm{e}^{-\mathrm{i}\Delta}\theta_2\left(\frac{-\mathrm{i}u}{2K}\right)\theta_0\left(\frac{\chi - \mathrm{i}u}{2K}\right)\right]\Big/\theta_0\left(\frac{\chi}{2K}\right), \\ s_2 &= C\,\mathrm{e}^{\mathrm{i}\lambda z/2 - \mathrm{i}\lambda^2\bar{z}/2}, \end{aligned} \tag{7}$$

where M, N, C are arbitrary complex numbers and $\Delta = -\gamma\bar{z} + \beta\chi$ with

$$\begin{aligned} \gamma &= -\frac{p^2}{2}\left[\mathrm{dn}^2(u, k') + \frac{\mathrm{cn}^2(u, k')}{\mathrm{sn}^2(u, k')}\right], \\ \beta &= \frac{\mathrm{d}}{\mathrm{d}u}\ln\theta_0\left(\frac{\mathrm{i}u}{2K}\right) + \frac{1}{2}\frac{\mathrm{dn}(u, k')\mathrm{cn}(u, k')}{\mathrm{sn}(u, k')} + \frac{\mathrm{sn}(u, k')\,\mathrm{dn}(u, k')}{\mathrm{cn}(u, k')}, \end{aligned} \tag{8}$$

and K and E are complete elliptic integrals of the first and the second kinds, respectively. In the above formula, a complex parameter u is related to the DT parameter λ as follows:

$$\lambda = \frac{v}{2} + p\frac{\mathrm{dn}(u, k')\mathrm{cn}(u, k')}{\mathrm{sn}(u, k')}. \tag{9}$$

Sym first introduced this solution in the description of vortex motion in hydrodynamics [30]. It was then applied to an NLSE-related problem in [31]. The proof of Sym's solution is given in appendix A. Finally, solutions of the case $\sigma = -1$ are obtained by substituting $p \to \mathrm{i}p$, $k \to \mathrm{i}k$ in equation (7).

[1] When we substitute $p \to \mathrm{i}p$ only, it becomes a *singular* solution of the defocusing CNLS equation.

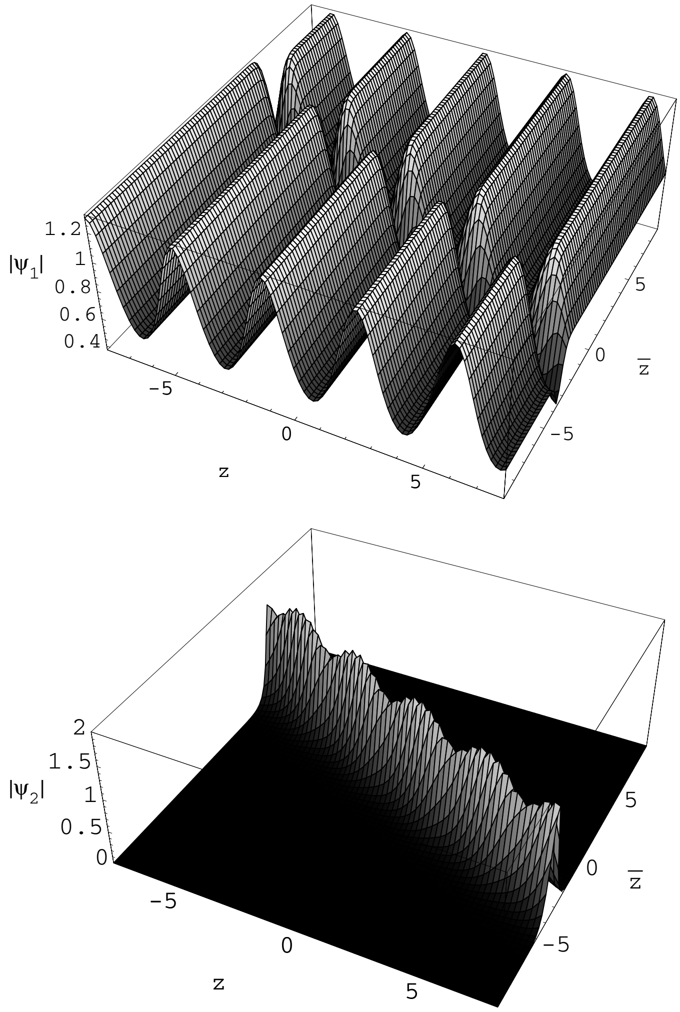

Figure 1. Dark–bright soliton pair of CNLS equation: $|\psi_1|$ shows a dark soliton residing on a cnoidal background. $|\psi_2|$ shows a bright soliton. The parameters are $v = 0, k = 0.9, p = 1.3, u = -0.38 + 0.63\mathrm{i}, M = 1, N = 0, C = 0.3$.

3. Solutions of self-focusing medium

3.1. A soliton crossing the cnoidal wave

Figure 1 shows (dark soliton+cnoidal wave) $|\psi_1|$, and a bright soliton $|\psi_2|$. It is obtained using equations (5)–(9) and taking $N = 0$. Other parameters for the figure are $v = 0, k = 0.9, p = 1.3, u = -0.38 + 0.63\mathrm{i}, M = 1, C = 0.3$. The figure is drawn using MATHEMATICA, which is also used in checking that the solution indeed satisfies the equation of motion (1). It shows the characteristic dark–bright pair soliton of CNLS equation where the dark soliton resides on a cnoidal background. This case becomes the well-known dark–bright soliton pair when we take $k = 0$ [6].

The dark–bright pair soliton moves along a line which satisfies $|s_2| \sim \left|s_0^{(M)}\right| \sim \left|s_1^{(M)}\right|$ in equation (7). Here, $\left|s_0^{(M)}\right|$ or $\left|s_1^{(M)}\right|$ mean M-coupled terms in equation (7). Thus $\left|s_0^{(M)}\right|$ is obtained from $|s_0|$ in equation (7) by taking $M = 1, N = 0$. In the subsequent section, we use notations $\left|s_0^{(N)}\right|$ or $\left|s_1^{(N)}\right|$, which means N-coupled terms in equation (7). In this case, $\left|s_0^{(N)}\right|$ is obtained from $|s_0|$ in equation (7) by taking $M = 0, N = 1$. As we move away from the soliton, it becomes $|s_2| \gg \left|s_0^{(M)}\right| \sim \left|s_1^{(M)}\right|$ or $|s_2| \ll \left|s_0^{(M)}\right| \sim \left|s_1^{(M)}\right|$, and ψ_2 in equation (6) becomes zero. We can see from equation (7) that the dominating factor of $\left|s_0^{(M)}\right|$ (or $|s_1^{(M)}|$) is $\exp(-\mathrm{Im}\,\Delta)$. Other terms give very small oscillating behaviour. The dominating factor of $|s_2|$ is $\exp(\mathrm{Im}(\lambda^2\bar{z} - \lambda z)/2)$. Thus, the soliton pair moves along the line $z = \alpha\bar{z}$, where

$$\alpha = \mathrm{Im}(\lambda^2/2 - \gamma + pv\beta)/\mathrm{Im}(\beta p + \lambda/2). \tag{10}$$

The value α of the soliton for parameters of figure 1 is calculated to be -1.92 using equation (10), which is in accordance with figure 1.

Another interesting feature of figure 1 ($|\psi_1|$) is that the crest of the cnoidal background shifts constantly across the dark soliton. The shift is calculated as following. As we move away from the soliton such that $|s_2| \gg \left|s_0^{(M)}\right| \sim \left|s_1^{(M)}\right|$, we find that $\psi_1^{c-s}(z,\bar{z}) \to \psi_1^c(z,\bar{z})$ from equation (6). Thus $\left|\psi_1^{c-s}\right| \to p\,\mathrm{dn}\chi$ in this region. On the other side of the dark soliton, it becomes $|s_2| \ll \left|s_0^{(M)}\right| \sim \left|s_1^{(M)}\right|$, and $\psi_1^{c-s}(z,\bar{z}) \to \psi_1^c(z,\bar{z}) + \mathrm{i}(\lambda - \lambda^*)\big(s_1^{(M)}/s_0^{(M)} + s_0^{(M)*}/s_1^{(M)*}\big)^{-1}$. Using that $s_1^{(M)}/s_0^{(M)} = -\mathrm{e}^{-\mathrm{i}\zeta}\,\mathrm{sn}(-\mathrm{i}u)\,\mathrm{dn}(\chi + \mathrm{i}u)/\mathrm{cn}(-\mathrm{i}u)$. (Using equations (7) and (A.4) with $N = 0$), we find that (for simplicity, we take $u = \mathrm{i}u_I$)

$$\psi_1^{c-s}\,\mathrm{e}^{-\mathrm{i}\zeta} \to p\,\mathrm{dn}\chi - 2p\frac{\mathrm{dn}\,u_I}{\mathrm{cn}\,u_I\mathrm{sn}\,u_I}\left(\frac{\mathrm{sn}\,u_I\,\mathrm{dn}(\chi - u_I)}{\mathrm{cn}\,u_I} + \frac{\mathrm{cn}\,u_I}{\mathrm{sn}\,u_I\,\mathrm{dn}(\chi - u_I)}\right)^{-1}, \tag{11}$$

where we use equations (9), (A.5) and $\mathrm{sn}\,u_I \equiv \mathrm{sn}(u_I, k)$. Using the addition theorem, equation (11) can be written as

$$\begin{aligned}
p&\frac{\mathrm{dn}(\chi - u_I)\,\mathrm{dn}\,u_I - k^2\mathrm{sn}\,u_I\mathrm{cn}\,u_I\mathrm{sn}(\chi - u_I)\mathrm{cn}(\chi - u_I)}{1 - k^2\mathrm{sn}^2u_I\mathrm{sn}^2(\chi - u_I)} - 2p\frac{\mathrm{dn}\,u_I\,\mathrm{dn}(\chi - u_I)}{\mathrm{sn}^2u_I\,\mathrm{dn}^2(\chi - u_I) + \mathrm{cn}^2u_I}\\
&= p\frac{-\mathrm{dn}(\chi - u_I)\,\mathrm{dn}\,u_I - k^2\mathrm{sn}\,u_I\mathrm{cn}\,u_I\mathrm{sn}(\chi - u_I)\,\mathrm{cn}(\chi - u_I)}{1 - k^2\mathrm{sn}^2u_I\mathrm{sn}^2(\chi - u_I)}\\
&= -p\,\mathrm{dn}(\chi - 2u_I).
\end{aligned} \tag{12}$$

Thus, $\left|\psi_1^{c-s}\right| \to p\,\mathrm{dn}(\chi - 2u_I)$ in this region, and the shift of crest is $2u_I$. For complex values of $u = u_R + \mathrm{i}u_I$, we find the following identity which is checked numerically for various values of complex u and real χ, k:

$$\begin{aligned}
&\left|\mathrm{dn}(\chi, k) - \left(\frac{\mathrm{dn}(-\mathrm{i}u, k)}{\mathrm{cn}(-\mathrm{i}u, k)\mathrm{sn}(-\mathrm{i}u, k)} + \frac{\mathrm{dn}(\mathrm{i}u^*, k)}{\mathrm{cn}(\mathrm{i}u^*, k)\mathrm{sn}(\mathrm{i}u^*, k)}\right)\right.\\
&\qquad\left.\times\left(\frac{\mathrm{sn}(-\mathrm{i}u, k)\,\mathrm{dn}(\chi + \mathrm{i}u, k)}{\mathrm{cn}(-\mathrm{i}u, k)} + \frac{\mathrm{cn}(\mathrm{i}u^*, k)}{\mathrm{sn}(\mathrm{i}u^*, k)\,\mathrm{dn}(\chi - \mathrm{i}u^*, k)}\right)^{-1}\right|\\
&\quad= \mathrm{dn}(\chi - 2u_I, k).
\end{aligned} \tag{13}$$

Using equation (13), we find that $\left|\psi_1^{c-s}\right| \to p\,\mathrm{dn}(\chi - 2u_I) = p\,\mathrm{dn}(\chi - 2\,\mathrm{Im}\,u)$ still holds. We see that the parameter u is suitable (compared to the DT parameter λ) in describing the characteristics of (soliton+cnoidal) system. The shift of the crest in terms of z is $z \to z + 2\,\mathrm{Im}\,u/p$ when we take $v = 0$ in $\chi = -p(z - v\bar{z})$. In figure 1, this shift is 0.97.

3.2. Soliton on top of cnoidal wave

Recently there arose new interest in optically-induced lattices and the localization of light in those gratings [16–18]. Especially, [18] describes a stationary configuration of (soliton+cnoidal wave) system where a soliton moves in parallel with the crest of a cnoidal wave. In this case, the velocity of the soliton is zero and equation (10) requires that $\lambda^2/2 - \gamma + pv\beta$ should be a real number. This, in turn, requires that u takes forms $u = \mathrm{i}u_I$ or $u = \mathrm{i}u_I + K'$. To see this, use equations (8), (9) and (A.5). Note that [32]

$$\mathrm{sn}(a + \mathrm{i}K') = \frac{1}{\mathrm{sn}\, a} \qquad \mathrm{cn}(a + \mathrm{i}K') = -\frac{\mathrm{i}}{k}\frac{\mathrm{dn}\, a}{\mathrm{sn}\, a} \qquad \mathrm{dn}(a + \mathrm{i}K') = -\mathrm{i}\frac{\mathrm{cn}\, a}{\mathrm{sn}\, a}. \tag{14}$$

Here u_I is a real number and $K' = K(\sqrt{1 - k^2})$.

Figure 2 shows one such case. Here solitons run in parallel with the crest of the cnoidal wave. For this figure, we take $M = 1.1$, $N = 1$. In this case, there arises interference between the M and N-coupled terms in s_0, s_1 of equation (7). This interference results in an oscillating behaviour of solitons as shown in figure 2.

The oscillating behaviour disappears in the case of $N = 0$ (or $M = 0$), and we get a stationary configuration of (soliton+cnoidal wave) system. This case corresponds to the system discussed in [18]. (They use numerical analysis.) Various stationary configurations dealt in [18] can be obtained from our result in equations (6) and (7) by taking proper values of parameters. It includes cases taking $k \geqslant 1$, $0 \geqslant k \geqslant 1$, as well as taking $u = \mathrm{i}u_I$ or $u = \mathrm{i}u_I + K'$ with $N = 0$ (or $M = 0$). More detailed characteristics of solutions corresponding to this specific choice of parameter are discussed elsewhere [33].

3.3. Soliton fusion on a cnoidal wave background

A more general solution describing (soliton+cnoidal wave) is given by taking $M \neq 0$, $N \neq 0$ in equation (7). Figure 3 describes one of these cases. We see that a dark–bright pair breaks up into another dark–bright pair plus an oscillating soliton. This feature was first found in [5]. Our result is a generalization of the result in [5], such that a dark soliton resides on a cnoidal wave instead on a constant background.

There are three soliton lines in figure 3. The direction of these lines is calculated similarly as in section 3.1. The directions of two dark–bright pairs are given by lines satisfying either $|s_2| \sim |s_0^{(M)}| \sim |s_1^{(M)}|$ or $|s_2| \sim |s_0^{(N)}| \sim |s_1^{(N)}|$. The direction of soliton line for $|s_2| \sim |s_0^{(M)}| \sim |s_1^{(M)}|$ is given in equation (10) of section 3.1. It was $z = -1.92\bar{z}$ for parameters of figure 1 (or figure 3). In contrast to the case of figure 1 (where the soliton exists on a full straight line), the soliton in figure 3 (which exists on a half of the line) exists only in the region satisfying $|s_2| \sim |s_0^{(M)}| \sim |s_1^{(M)}| \gg |s_0^{(N)}| \sim |s_1^{(N)}|$. Using the expression for ψ_2 in equation (6), it is easy to see that the soliton disappears in the region $|s_2| \sim |s_0^{(M)}| \sim |s_1^{(M)}| \ll |s_0^{(N)}| \sim |s_1^{(N)}|$.

Another dark–bright pair moves along a line satisfying $|s_2| \sim |s_0^{(N)}| \sim |s_1^{(N)}|$, or $z = \hat{\alpha}\bar{z}$, where

$$\hat{\alpha} = \frac{\mathrm{Im}(\lambda^2/2 + \gamma - pv\beta)}{\mathrm{Im}(-\beta p + \lambda/2)}. \tag{15}$$

In this case, the dominating factor of $|s_0^{(N)}|$ (or $|s_1^{(N)}|$) in equation (7) is $\exp(\mathrm{Im}\,\Delta)$. The dominating factor of $|s_2|$ is still $\exp(\mathrm{Im}(\lambda^2\bar{z} - \lambda z)/2)$. Thus, the soliton pair moves along the line $\mathrm{Im}\,\Delta = \mathrm{Im}(\lambda^2\bar{z} - \lambda z)/2)$, which gives equation (15). For parameters of figure 3, $\hat{\alpha} = 0.44$.

The oscillating soliton in $|\psi_1|$ of figure 3 moves along a line $z = \tilde{\alpha}\bar{z}$, where

$$\tilde{\alpha} = \frac{\mathrm{Im}(-\gamma + pv\beta)}{\mathrm{Im}(\beta p)}. \tag{16}$$

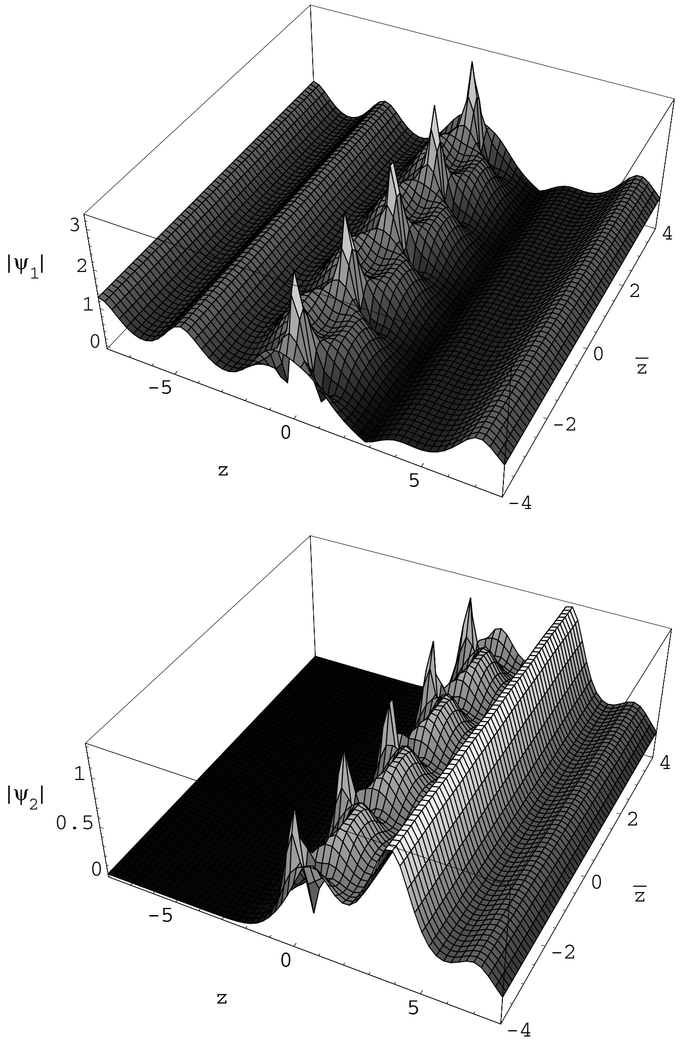

Figure 2. Dark–bright soliton pair moving parallely along the crest of cnoidal wave: $|\psi_1|$ for dark soliton, $|\psi_2|$ for bright soliton. The parameters are $v = 0, k = 0.9, p = 1.3, u = 0.63\mathrm{i}, M = 1.1, N = 1, C = 0.3$.

In this case, the line is described by $|s_0^{(M)}|(\sim|s_1^{(M)}|) \sim |s_0^{(N)}|(\sim|s_1^{(N)}|)$. The dominating factor of $|s_0^{(M)}|$ (or $|s_1^{(M)}|$) is $\exp(-\mathrm{Im}\,\Delta)$, while that of $|s_0^{(N)}|$ (or $|s_1^{(N)}|$) is $\exp(\mathrm{Im}\,\Delta)$. Thus, the oscillating soliton moves along a line $\mathrm{Im}\,\Delta = 0$, which gives equation (16). For parameters of figure 3, $\tilde{\alpha} = -2.89$. Note that on a line of the oscillating soliton, it becomes $|s_2| \ll |s_0| \sim |s_1|$. Thus there appears no soliton in $|\psi_2|$ along this line (see equation (6)). The nature of the oscillating soliton in ψ_1 is exactly that of the single-component NLSE in [31]. In the region of $|s_2| \gg |s_0| \sim |s_1|$, the oscillating soliton disappears even in $|\psi_1|$.

The shift of the crest of the cnoidal wave across the solitons can be similarly calculated as in section 3.1. Especially in the region (region I) where $|s_2| \gg |s_0^{(M)}| \sim |s_1^{(M)}|$ and $|s_2| \gg |s_0^{(N)}| \sim |s_1^{(N)}|$, we have $|\psi_1^{c-s}(z,\bar{z})| \to |\psi_1^c(z,\bar{z})| = p\,\mathrm{dn}(\chi)$. In the region (region II) where $|s_2| \ll |s_0^{(M)}| \sim |s_1^{(M)}|$ and $|s_0^{(N)}| \sim |s_1^{(N)}| \ll |s_0^{(M)}|$

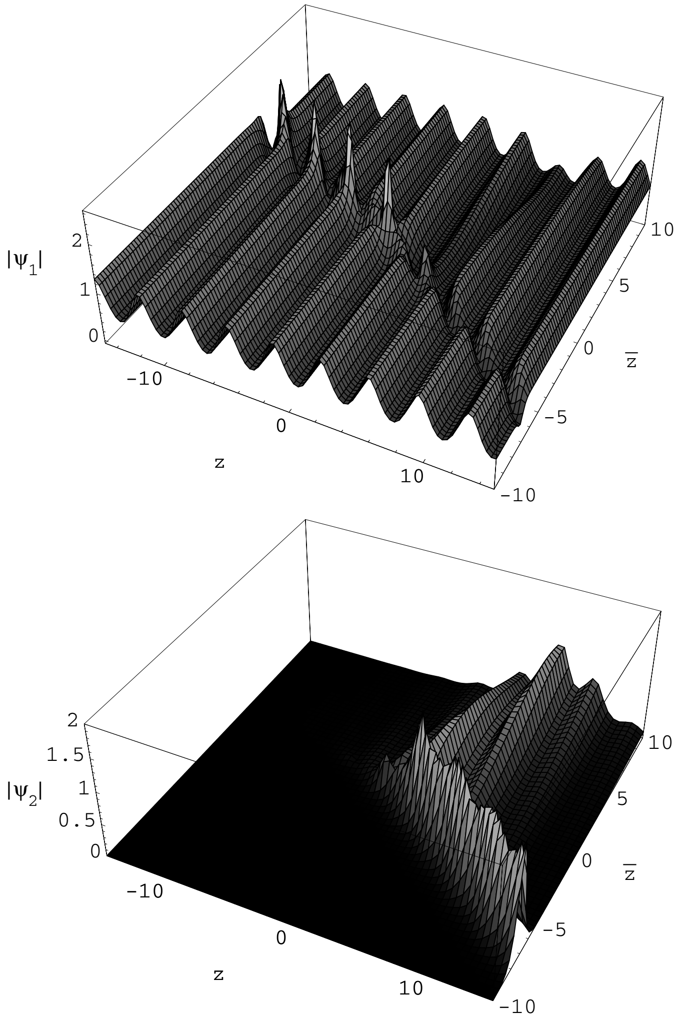

Figure 3. Dark–bright soliton pair breaks up into another dark–bright pair plus an oscillating soliton: $|\psi_1|$ shows a dark soliton breaking up into dark plus oscillating solitons. $|\psi_2|$ shows a bright soliton changing its direction. The parameters are $v = 0, k = 0.9, p = 1.3, u = -0.38 + 0.63\mathrm{i}, M = 1.1, N = 1, C = 0.3$.

$\sim \left|s_1^{(M)}\right|$, we have $\psi_1^{c-s}(z,\bar{z}) \rightarrow \psi_1^c(z,\bar{z}) + \mathrm{i}(\lambda - \lambda^*)\big(s_1^{(M)}\big/s_0^{(M)} + s_0^{(M)*}\big/s_1^{(M)*}\big)^{-1}$ and $\left|\psi_1^{c-s}\right| \rightarrow p\,\mathrm{dn}(\chi - 2u_I) = p\,\mathrm{dn}(\chi - 2\,\mathrm{Im}\,u)$, see section 3.1. In the region (region III) where $|s_2| \ll \left|s_0^{(N)}\right| \sim \left|s_1^{(N)}\right|$ and $\left|s_0^{(N)}\right| \sim \left|s_1^{(N)}\right| \gg \left|s_0^{(M)}\right| \sim \left|s_1^{(M)}\right|$, we have $\psi_1^{c-s}(z,\bar{z}) \rightarrow \psi_1^c(z,\bar{z}) + \mathrm{i}(\lambda - \lambda^*)\big(s_1^{(N)}\big/s_0^{(N)} + s_0^{(N)*}\big/s_1^{(N)*}\big)^{-1}$. In this case, $s_1^{(N)}\big/s_0^{(N)} = -\mathrm{e}^{-\mathrm{i}\zeta}\,\mathrm{cn}(-\mathrm{i}u)/(\mathrm{dn}(\chi - iu)\,\mathrm{sn}(-\mathrm{i}u))$. Similar procedure used in equations (11) and (12) gives us $\left|\psi_1^{c-s}\right| \rightarrow p\,\mathrm{dn}(\chi + 2u_I) = p\,\mathrm{dn}(\chi + 2\,\mathrm{Im}\,u)$.

Regions I and II meet at a boundary, along which the dark–bright soliton pair moves with the velocity α in equation (10). Along the boundary between regions I and III, the soliton pair moves with the velocity $\hat{\alpha}$ in equation (15). Finally, the oscillating soliton moves along the boundary between regions II and III with the velocity $\tilde{\alpha}$ in equation (16). Thus the relative shift of the crest of the cnoidal wave across the boundary described by $\tilde{\alpha}$ is given by

$\chi \to \chi - 4\,\mathrm{Im}\,u$. Then the shift of the crest in terms of z is $z \to z + 4\,\mathrm{Im}\,u/p$ when we take $v = 0$ in $\chi = -p(z - v\bar{z})$. For parameters of figure 3, this shift is 1.94.

4. Solution of defocusing medium

Figure 4 shows a dark soliton on a cnoidal background in a defocusing medium. It is obtained using equations (5), (6) and (7) and taking $\sigma = -1$, $v = 0, k = 0.9\mathrm{i}$, $p = 1.3\mathrm{i}, u = -0.3 - 0.98\mathrm{i}, M = 1, N = 0, C = 0.1$. Note that in the case of defocusing medium ($\sigma = -1$), we take imaginary k, p. Another specific feature of the defocusing medium is that it is not permitted to take both M and N to be simultaneously nonzero. This fact is explained as follows. The denominator part of equation (7) is $|s_0|^2 - |s_1|^2 - |s_2|^2$, which is required to be positive definite or negative definite on all $(z, \bar{z})$ plane. Otherwise, we will get an unphysical singular solution. But there always exists a region in the $(z, \bar{z})$ plane where $|s_2| \gg |s_0|$, which means the denominator must be negative definite. Thus we need $|s_0|^2 - |s_1|^2$ should be negative definite. In the region $\mathrm{Im}\,\Delta \sim -\infty$ where $|s_0| \sim \left|s_0^{(M)}\right|, |s_1| \sim \left|s_1^{(M)}\right|$, the negative definiteness requires $1 < \left|s_1^{(M)}\big/s_0^{(M)}\right| = |\mathrm{sn}(-\mathrm{i}u)\,\mathrm{dn}(\chi + \mathrm{i}u)/\mathrm{cn}(-\mathrm{i}u)|$. On the other hand, in the region $\mathrm{Im}\,\Delta \sim \infty$ where $|s_0| \sim \left|s_0^{(N)}\right|, |s_1| \sim \left|s_1^{(N)}\right|$, we need $1 < \left|s_1^{(N)}\big/s_0^{(N)}\right| = |\mathrm{cn}(-\mathrm{i}u)/(\mathrm{dn}(\chi - \mathrm{i}u)\mathrm{sn}(-\mathrm{i}u))|$. As $|\mathrm{dn}(\chi \pm \mathrm{i}u)|$ is an oscillating function of χ, the negative definiteness from the two regions results in a contradiction on the magnitude of $|\mathrm{sn}(-\mathrm{i}u)/\mathrm{cn}(-\mathrm{i}u)|$. Thus we need that one of M, N should be zero. Thus there does not exist the phenomenon of soliton fusion in the case of the defocusing medium.

When we take $N = 0$ as in figure 4, we need $1 < |\mathrm{sn}(-\mathrm{i}u)\,\mathrm{dn}(\chi + \mathrm{i}u)/\mathrm{cn}(-\mathrm{i}u))|$. In appendix B, we prove that

$$(1 + k^2)^{(1/4)} \left|\frac{\mathrm{sn}(-\mathrm{i}u_R + m\tilde{K}', \mathrm{i}k)}{\mathrm{cn}(-\mathrm{i}u_R + m\tilde{K}', \mathrm{i}k)}\right| = |\mathrm{dn}(\chi + \mathrm{i}u_R - m\tilde{K}', \mathrm{i}k)|/(1 + k^2)^{(1/4)} = 1, \tag{17}$$

where

$$\tilde{K}' = \frac{K'(1/\sqrt{1 + k^2})}{2\sqrt{1 + k^2}}, \tag{18}$$

m is an odd number ($m = 2n + 1$), $\chi = -\mathrm{i}p(z - v\bar{z})$ and k, p, v, u_R are real numbers. (Here, we use the fact that $k \to \mathrm{i}k$, $p \to \mathrm{i}p$ in the defocusing medium.) We also found that

$$(1 + k^2)^{(1/4)} \left|\frac{\mathrm{sn}(-\mathrm{i}u_R + u_I, \mathrm{i}k)}{\mathrm{cn}(-\mathrm{i}u_R + u_I, \mathrm{i}k)}\right| > 1 \qquad |\mathrm{dn}(\chi + \mathrm{i}u_R - u_I, \mathrm{i}k)|/(1 + k^2)^{(1/4)} > 1, \tag{19}$$

when

$$(4n + 1)\tilde{K}' < u_I < (4n + 3)\tilde{K}' \tag{20}$$

where n is an arbitrary integer. On the outsides of the interval in equation (20), equation (19) reverses its inequality sign. Thus we can see that the negative definiteness constrains the possible value of u_I to the intervals given in equation (20). Similarly we must take the value u_I in the outsides of intervals in equation (19) when we take $M = 0$. For the parameters of figure 4, $\tilde{K}' = 0.67$.

The dark–bright pair soliton moves along a line, which satisfies $|s_2| \sim \left|s_0^{(M)}\right| \sim \left|s_1^{(M)}\right|$ (the $N = 0$ case). Thus we can use equation (10) to describe the direction of the soliton line. In the case of defocusing medium, we must take imaginary p, k values. For parameters of figure 4, $\alpha = -2.48$. The soliton line shifts along the $\bar{z}$-axis (or z-axis) according to the relative magnitude of M and C. This shift is rather large in figure 4 compared to the other figures, which is due to the large $M/C = 10$ value.

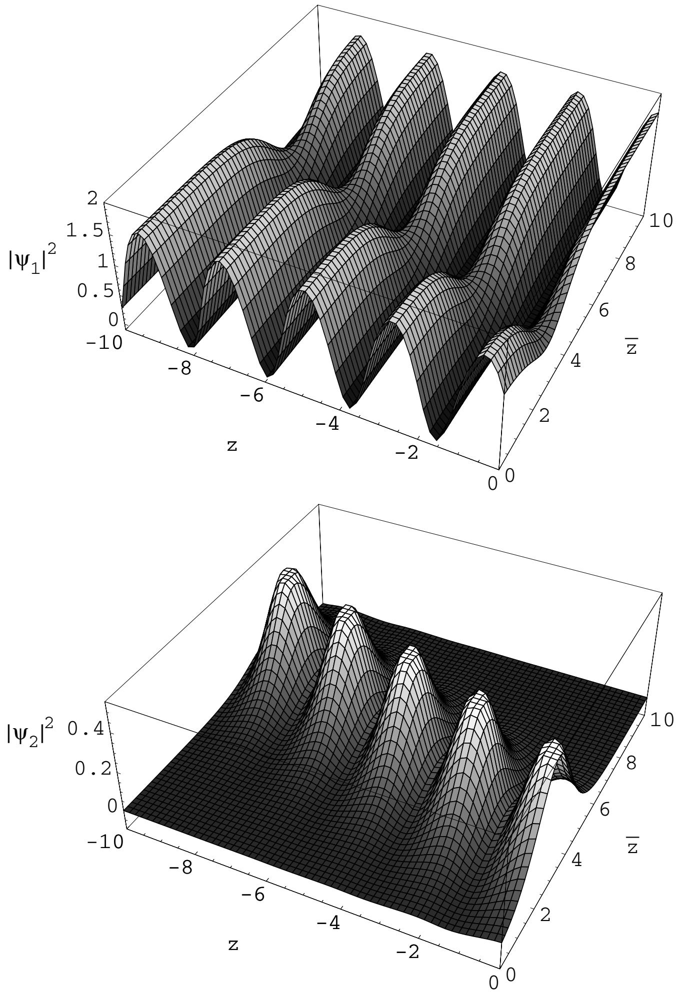

Figure 4. Dark–bright soliton pair moving in a defocusing medium: $|\psi_1|^2$, a dark soliton and $|\psi_2|^2$, a bright soliton. The parameters are $v = 0, k = 0.9\mathrm{i}$, $p = 1.3\mathrm{i}, u = -0.3 - 0.98\mathrm{i}, M = 1, N = 0, C = 0.1$.

The shift of the crest for the case of the defocusing medium is similarly calculated as in section 3.1. At the region $|s_2| \gg \left|s_0^{(M)}\right|$, $\psi_1^{c-s}(-p\mathrm{i}(z - v\bar{z})) = \psi_1^c(-p\mathrm{i}(z - v\bar{z}))$. To calculate ψ_1^{c-s} in the region $|s_2| \ll \left|s_0^{(M)}\right|$, we use the following formula

$$\begin{aligned}
&\left|\mathrm{dn}(\mathrm{i}\chi, \mathrm{i}k) - \left(\frac{\mathrm{dn}(-\mathrm{i}u, \mathrm{i}k)}{\mathrm{cn}(-\mathrm{i}u, \mathrm{i}k)\mathrm{sn}(-\mathrm{i}u, \mathrm{i}k)} - \frac{\mathrm{dn}(\mathrm{i}u^*, -\mathrm{i}k)}{\mathrm{cn}(\mathrm{i}u^*, -\mathrm{i}k)\mathrm{sn}(\mathrm{i}u^*, -\mathrm{i}k)}\right)\right. \\
&\qquad\left. \times \left(\frac{\mathrm{sn}(-\mathrm{i}u, \mathrm{i}k)\,\mathrm{dn}(\mathrm{i}\chi + \mathrm{i}u, \mathrm{i}k)}{\mathrm{cn}(-\mathrm{i}u, \mathrm{i}k)} - \frac{\mathrm{cn}(\mathrm{i}u^*, -\mathrm{i}k)}{\mathrm{sn}(\mathrm{i}u^*, -\mathrm{i}k)\,\mathrm{dn}(-\mathrm{i}\chi - \mathrm{i}u^*, -\mathrm{i}k)}\right)^{-1}\right| \\
&\quad = |\mathrm{dn}(\mathrm{i}\chi + 2\mathrm{i}\,\mathrm{Re}\,u, \mathrm{i}k)|,
\end{aligned} \tag{21}$$

where u is complex and χ, k are real. Then using equations (6) and (21), we find that $\left|\psi_1^{c-s}(-p\mathrm{i}(z - v\bar{z}))\right| = \left|\psi_1^c(-p\mathrm{i}(z - v\bar{z}) + 2\mathrm{i}\,\mathrm{Re}\,u)\right|$ for real p. The shift of the crest in terms of z is $z \to z - 2\,\mathrm{Re}\,u/p$ for the case of $v = 0$. In figure 4, this shift is 0.46.

5. Conclusions

In this paper, we have introduced (soliton+cnoidal wave) solutions of the CNLS equation. They were obtained using the DT and introducing a solution of the associated linear problem on a cnoidal background. We calculate the moving direction of a soliton on a cnoidal background and the shift of the crest of a cnoidal wave. We also found a solution in the self-focusing case where a dark–bright pair breaks up into another dark–bright pair and an oscillating soliton. These type of solutions, which can be easily applicable to the analysis of physically interesting processes, seem rather rare in the literature of physics. These solutions can be used, for example, in describing the localized states in optically induced refractive index gratings.

The stability analysis of these solutions is retained for future study. In fact, there have already appeared some numerical studies on this subject [18]. There it was shown that the (soliton+cnoidal wave) system is unstable or weakly stable for the focusing case, while it is stable for the defocusing case.

Acknowledgments

I am grateful to Yuri S Kivshar for showing me the paper [18] prior to publication. This work was supported by Korea Research Foundation Grant (KRF-2003-070-C00011).

Appendix A. Proof of Sym's solution

In this appendix, we show that s_0, s_1, s_2 in equation (7) indeed satisfy the linear equation (4). Consider the following equation, which is obtained from the ∂_z-part of equation (4);

$$\partial_z s_0 + \psi_1^c s_1 + \mathrm{i}\lambda s_0/2 = 0. \tag{A.1}$$

By inserting s_0, s_1 (for simplicity, we consider a case of $M = 1$, $N = 0$ in equation (7)) into equation (A.1) and taking ψ_1^c as in equation (5), we get

$$-\mathrm{i}p\beta - \frac{p}{2K}\frac{\theta_0'\left(\frac{\chi+\mathrm{i}u}{2K}\right)}{\theta_0\left(\frac{\chi+\mathrm{i}u}{2K}\right)} + \frac{p}{2K}\frac{\theta_0'\left(\frac{\chi}{2K}\right)}{\theta_0\left(\frac{\chi}{2K}\right)} - p\,\mathrm{dn}(\chi, k)\frac{\theta_1\left(\frac{-\mathrm{i}u}{2K}\right)}{\theta_2\left(\frac{-\mathrm{i}u}{2K}\right)}\frac{\theta_3\left(\frac{\chi+\mathrm{i}u}{2K}\right)}{\theta_0\left(\frac{\chi+\mathrm{i}u}{2K}\right)} + p\frac{\mathrm{dn}(u, k')\,\mathrm{cn}(u, k')}{\mathrm{sn}(u, k')}. \tag{A.2}$$

Inserting β in equation (8) into equation (A.2) and using the following identities [26, 32],

$$\int_0^u \mathrm{dn}^2 u\,\mathrm{d}u = \frac{1}{2K}\frac{\theta_0'\left(\frac{u}{2K}\right)}{\theta_0\left(\frac{u}{2K}\right)} + \frac{E}{K}u, \tag{A.3}$$

$$\mathrm{sn}\,u = \frac{1}{\sqrt{k}}\frac{\theta_1\left(\frac{u}{2K}\right)}{\theta_0\left(\frac{u}{2K}\right)}, \qquad \mathrm{cn}\,u = \frac{\sqrt{k'}}{\sqrt{k}}\frac{\theta_2\left(\frac{u}{2K}\right)}{\theta_0\left(\frac{u}{2K}\right)}, \qquad \mathrm{dn}\,u = \sqrt{k'}\frac{\theta_3\left(\frac{u}{2K}\right)}{\theta_0\left(\frac{u}{2K}\right)}, \tag{A.4}$$

$$\mathrm{sn}(\mathrm{i}u, k') = \mathrm{i}\frac{\mathrm{sn}(u, k)}{\mathrm{cn}(u, k)}, \qquad \mathrm{cn}(\mathrm{i}u, k') = \frac{1}{\mathrm{cn}(u, k)}, \qquad \mathrm{dn}(\mathrm{i}u, k') = \frac{\mathrm{dn}(u, k)}{\mathrm{cn}(u, k)}, \tag{A.5}$$

we get

$$-p\int_0^{\chi+\mathrm{i}u}[\mathrm{dn}^2 v - \mathrm{dn}^2(v - \mathrm{i}u)]\,\mathrm{d}v + \frac{\mathrm{sn}(-\mathrm{i}u, k)}{\mathrm{cn}(-\mathrm{i}u, k)}[\mathrm{dn}(-\mathrm{i}u, k) - \mathrm{dn}(\chi, k)\,\mathrm{dn}(\chi + \mathrm{i}u, k)]. \tag{A.6}$$

Finally, using the following identity (it is a result of the addition theorem of Jacobi's elliptic functions),

$$\mathrm{dn}^2(a-b) - \mathrm{dn}^2 a = k^2 \frac{\mathrm{sn}\, b}{\mathrm{cn}\, b}[\mathrm{cn}\, a\, \mathrm{sn}\, a\, \mathrm{dn}(a-b) + \mathrm{dn}\, a\, \mathrm{cn}(a-b)\, \mathrm{sn}(a-b)]$$
$$= -\frac{\mathrm{sn}\, b}{\mathrm{cn}\, b}\frac{\mathrm{d}}{\mathrm{d}a}[\mathrm{dn}\, a\, \mathrm{dn}(a-b)], \tag{A.7}$$

we can see that equation (A.6) becomes zero. The other ∂_z-part of the linear equation (4) (including the $M = 0$, $N = 1$ case of s_0, s_1 in equation (7)) is similarly proved.

A $\partial_{\bar{z}}$-part of equation (4) is (the $\sigma = 1$ case)

$$\partial_{\bar{z}} s_0 + \mathrm{i}\left|\psi_1^c\right|^2 s_0 - \mathrm{i}\partial_z \psi_1^c s_1 - \lambda \psi_1^c s_1 - \mathrm{i}\lambda^2 s_0/2 = 0. \tag{A.8}$$

By inserting s_0, s_1 (for simplicity, we take $M = 1$, $N = 0$ in equation (7)) into the left part of equation (A.8) and taking ψ_1^c as in equation (5), we get

$$\frac{\mathrm{i}}{2}\left[-p^2(2-k^2) + \frac{v^2}{4}\right] - \mathrm{i}\gamma - v\left(\frac{\partial_z s_0}{s_0} + \mathrm{i}\frac{v}{4}\right) + \mathrm{i}p^2\,\mathrm{dn}^2\chi - \frac{\mathrm{i}}{2}\lambda^2$$
$$+ p\left[\mathrm{i}\partial_z\,\mathrm{dn}\,\chi + \left(\lambda + \frac{v}{2}\right)\mathrm{dn}\,\chi\right]\frac{\theta_1\left(\frac{-\mathrm{i}u}{2K}\right)}{\theta_2\left(\frac{-\mathrm{i}u}{2K}\right)}\frac{\theta_3\left(\frac{\chi+\mathrm{i}u}{2K}\right)}{\theta_0\left(\frac{\chi+\mathrm{i}u}{2K}\right)}. \tag{A.9}$$

Using equation (A.1) and the identity (A.4), equation (A.9) becomes

$$-\frac{\mathrm{i}}{2}p^2(2-k^2) + \frac{\mathrm{i}}{8}v^2 + \mathrm{i}\frac{p^2}{2}\left[\mathrm{dn}^2(u,k') + \frac{\mathrm{cn}^2(u,k')}{\mathrm{sn}^2(u,k')}\right] + \frac{\mathrm{i}}{2}\lambda v - \frac{\mathrm{i}}{4}v^2 - \frac{\mathrm{i}}{2}\lambda^2$$
$$+\mathrm{i}p^2\,\mathrm{dn}^2\chi - \left(\frac{p}{2}v\,\mathrm{dn}\,\chi - \mathrm{i}p^2k^2\mathrm{sn}\,\chi\,\mathrm{cn}\,\chi - \lambda p\,\mathrm{dn}\,\chi\right)\frac{\mathrm{sn}(-\mathrm{i}u,k)}{\mathrm{cn}(-\mathrm{i}u,k)}\,\mathrm{dn}(\chi+\mathrm{i}u,k). \tag{A.10}$$

With the help of equations (A.5) and (9), repeated applications of addition theorem on equation (A.10) give zero, which proves equation (A.8). The other $\partial_{\bar{z}}$-part of the linear equation (4) (including the $M = 0$, $N = 1$ case of s_0, s_1 in equation (7)) is similarly proved.

Appendix B. Proof of equation (17)

We first start with

$$(1+k^2)^{(1/4)}\frac{\mathrm{sn}(-\mathrm{i}u,\mathrm{i}k)}{\mathrm{cn}(-\mathrm{i}u,\mathrm{i}k)} = \mathrm{i}\,\mathrm{sn}(-\sqrt{1+k^2}u,\tilde{k})/(1+k^2)^{(1/4)}, \tag{B.1}$$

where $\tilde{k} = 1/\sqrt{1+k^2}$ with k real and u is a complex value. By inserting $u = u_R + \mathrm{i}\tilde{K}'$ in equation (18) and using the addition theorem, equation (B.1) becomes

$$\frac{\mathrm{sn}(-\sqrt{1+k^2}u_R,\tilde{k})(1+\tilde{k}) - \mathrm{i}\,\mathrm{cn}(-\sqrt{1+k^2}u_R,\tilde{k})\,\mathrm{dn}(-\sqrt{1+k^2}u_R,\tilde{k})}{1+\tilde{k}\,\mathrm{sn}(-\sqrt{1+k^2}u_R,\tilde{k})}, \tag{B.2}$$

where we use

$$\mathrm{sn}(-\mathrm{i}\sqrt{1+k^2}\tilde{K}',\tilde{k}) = -\mathrm{i}(1+k^2)^{(1/4)} \qquad \mathrm{dn}(-\mathrm{i}\sqrt{1+k^2}\tilde{K}',\tilde{k}) = \sqrt{1+\tilde{k}},$$
$$\mathrm{cn}(-\mathrm{i}\sqrt{1+k^2}\tilde{K}',\tilde{k}) = (1+k^2)^{(1/4)}\sqrt{1+\tilde{k}}. \tag{B.3}$$

It is now easy to see that the absolute value of equation (B.2) is 1.

In the case of dn formula,

$$\mathrm{dn}(-\mathrm{i}u, \mathrm{i}k)/(1+k^2)^{(1/4)} = \frac{\mathrm{cn}(-\sqrt{1+k^2}u, \tilde{k})}{\mathrm{dn}(-\sqrt{1+k^2}u, \tilde{k})(1+k^2)^{(1/4)}}. \tag{B.4}$$

Applying a similar procedure used in obtaining equation (B.2) to equation (B.4) gives

$$\frac{\mathrm{cn}(-\sqrt{1+k^2}u_R, \tilde{k}) + \mathrm{i}\,\mathrm{sn}(-\sqrt{1+k^2}u_R, \tilde{k})\,\mathrm{dn}(-\sqrt{1+k^2}u_R, \tilde{k})}{\mathrm{dn}(-\sqrt{1+k^2}u_R, \tilde{k}) - \mathrm{i}\tilde{k}\,\mathrm{cn}(-\sqrt{1+k^2}u_R, \tilde{k})\,\mathrm{sn}(-\sqrt{1+k^2}u_R, \tilde{k})}, \tag{B.5}$$

whose absolute value is 1. It is easy to extend the above results to the case of $u = u_R - \mathrm{i}\tilde{K}'$. Using that $\mathrm{sn}(-\mathrm{i}u, \mathrm{i}k)/\mathrm{cn}(-\mathrm{i}u, \mathrm{i}k)$, $\mathrm{dn}(-\mathrm{i}u, \mathrm{i}k)$ are periodic functions in u_I with the period $2\tilde{K}'$, we can obtain the result in equation (17) in section 4.

To obtain the result shown in equation (19), we should go through similar procedures in the above paragraphs with much complex expressions in this case. Instead of going through a tedious calculation, we use MAPLE to numerically check the validity of equation (19).

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